Copy 1
math files

THE GAMBLER'S RUIN WITH SOFT HEARTED ADVERSARY

R. E. Fagen
T. A. Lehrer
MATH-3
24 January 1957
29 pages

The random walk on a one dimensional lattice with restraining barrier at the origin and absorbing barrier at N is discussed for the case where only steps of r units to the right and ℓ units to the left are allowed. Asymptotic expressions for the mean and variance of the duration of such a walk are derived, and, in some cases, the limiting distribution is found.

S-86 667

I. INTRODUCTION

The classical gambler's ruin problem, stated in terms of a random walk in one dimension, involves two absorbing barriers. If the walk is restricted to the non-negative integers, one can imagine the barriers placed at the origin and at some point $N > 0$. This problem has received much attention in the literature; for three different treatments see Feller [1], Uspensky [2], and Bartlett [4].

We consider here a random walk with an absorbing barrier at N and a restraining barrier at the origin. A restraining barrier, when reached, allows motion in one direction only; in our case, the restraining barrier at the origin prevents passage to the left, but has no effect on motion to the right. To be explicit, a particle reaching the origin would remain there as long as it tried to move to the left, but would be free to move to the right at any stage. (A particle attempting to move past the origin from the right would be allowed to move only as far as the origin.)

A random walk of the type described above corresponds to a gambler playing against an adversary who allows him to stay in the game even when broke. To our knowledge this problem is not treated in the literature; the similar problem with reflecting barrier has been considered in the references mentioned above, although not as extensively as the absorbing barrier problem. A restraining barrier is quite similar to a reflecting barrier, especially for large N, but since the results obtained here are not given explicitly elsewhere, it seems worthwhile to give a complete treatment of the former case.

In the following, we confine our attention to random walks on the non-negative integers i , $0 \leq i \leq N$, with restraining and absorbing barriers at the two end points 0 and N respectively. We further assume that the particle has, at any stage of the process, probability p of making a step r units to the right and probability $q = 1 - p$ of making a step ℓ units to the left, insofar as the barriers allow. The random variable of particular interest here is the duration of the walk (the number of steps the particle takes before it is absorbed at N) starting at the origin. In part II we give the generating function for the distribution of duration (and also exact expressions for the mean and variance) in the special case $r = \ell = 1$. Results quite similar to these are commonplace and can be found in any of the references cited above; they are given here for completeness. We also find limiting distributions for the duration, which, as far as we know, are not given explicitly elsewhere.

In part III we give an asymptotic expression for the mean duration with arbitrary steps r and ℓ . Although this type of asymptotic result is given for the case of two absorbing barriers [1], [2], [4], the methods used there do not apply to the present problem owing to the difference in boundary conditions. The only reference we know in which methods similar to those in part III are employed is Kemperman [5]; he too, however, is considering absorbing barriers, and treats a more general type of walk [steps of any integral size k , $-\ell \leq k \leq r$, permissible]. Our restriction to just two steps r and ℓ enables us to give simple explicit formulae.

Finally, in part IV, we discuss approximations which prove especially useful in the engineering of systems using this type of process [a consideration that has motivated this entire study], and also point out how the results for general steps r and ℓ are intuitively satisfying extensions of those for $r = \ell = 1$.

II. RANDOM WALK WITH UNIT STEPS

1. Generating Function of Duration.

We consider here the case in which the only possible steps are unit steps in either direction, so that $r = \ell = 1$. The particle starts from 0 and stops when it reaches the absorbing barrier at N for the first time. We let $p \neq 0$ and $q = 1 - p$ be the probabilities of steps to the right and left respectively.

In this case explicit expressions for the mean and variance of duration of the walk may be found as follows:

Let $u_{k,n}$ be the probability that the walk ends after exactly n steps, given that the particle is initially at position k ($n = 0, 1, \cdots$; $k = 0, 1, \cdots, N$). Note that $u_{k,n} = 0$ for $k < N - n$. The quantities $u_{k,n}$ satisfy the recurrence relations

$$(2.1) \quad u_{k,n+1} = p u_{k+1,n} + q u_{k-1,n} \qquad n = 0, 1, \cdots ; k = 1, \cdots, N-1$$

$$u_{0,n+1} = p u_{1,n} + q u_{0,n}$$

$$u_{N,n} = 0 \qquad n > 0$$

$$u_{N,0} = 1$$

Denote by $U_k(s)$ the generating function of the distribution $\{u_{k,n}\}$; that is

$$(2.2) \quad U_k(s) = \sum_{n=0}^{\infty} u_{k,n} s^n \qquad k = 0, \cdots, N \quad ,$$

We note first that

$$(2.3) \quad U_N(s) = 1 \ .$$

Since we are concerned with walks starting at the origin, the quantity of principal interest is $U_o(s)$, the generating function of the duration of a walk starting from 0 . As will be seen, the mean and variance of the duration can be obtained from $U_o'(1)$ and $U_o''(1)$.

Multiplying through by s^{n+1} and summing on n , we have, from (2.1) (noting that $u_{k,o} = 0$) ,

$$(2.4) \quad U_k(s) = p s U_{k+1}(s) + q s U_{k-1}(s) \qquad k = 1, \cdots, N-1$$

$$(2.5) \quad \begin{cases} U_o(s) = p s U_1(s) + q s U_o(s) \\ U_N(s) = 1 \end{cases}$$

Equation (2.4) is a second order difference equation in $U_k(s)$ for fixed s , with boundary conditions given by (2.5). The auxiliary equation associated with (2.4) is

(2.6) $$psx^2 - x + qs = 0$$

the roots of which are

(2.7) $$x_1, x_2 = (1/2ps)(1 \pm \sqrt{1 - 4pqs^2}) .$$

The solution of (2.4) is then of the form

(2.8) $$U_k = c_1 x_1^k + c_2 x_2^k$$

with c_1 and c_2 determined by (2.5). For $U_o(s)$ we find

(2.9) $$U_o(s) = \frac{2(2ps)^N\sqrt{1-4pqs^2}}{(1+\sqrt{1-4pqs^2})^N(1-2qs+\sqrt{1-4pqs^2}) - (1-\sqrt{1-4pqs^2})^N(1-2qs-\sqrt{1-4pqs^2})}$$

2. Mean Duration of the Walk $(p \neq q)$

We shall assume throughout sections 2 and 3 that $p \neq q$.

Let X_k denote the random variable representing the duration of the random walk when the particle starts at k , and let $D_k \equiv E(X_k)$ be the mean duration of this walk where $k = 0, 1, \cdots, N$. We observe that

(2.10) $$D_k = \sum_{n=0}^{\infty} n u_{k,n} = U_k'(1) .$$

Again we are ultimately interested only in $D_o = U_o'(1)$. This quantity may be obtained directly by differentiating the right member of (2.9) and setting $s = 1$; the following technique is somewhat more expedient, however, in addition to being more instructive.

Differentiating in (2.4) and (2.5) and setting $s = 1$ we have

$$(2.11) \qquad D_k = pD_{k+1} + qD_{k-1} + 1 \qquad k = 1, \cdots, N-1$$

$$(2.12) \qquad \begin{cases} D_o = pD_1 + qD_o + 1 \\ D_N = 0 \end{cases}$$

It is to be noted that (2.12) and (2.11) can be deduced directly from physical considerations. The argument is, roughly, that the expected duration starting from k is one more than the expected duration starting from the next position, which position will be either $k+1$ with probability p or $k-1$ with probability q.

Equation (2.11) is a non-homogenous recurrence relation; the solution of the associated linear homogeneous equation will be of the form $A_1 y_1^k + A_2 y_2^k$ where y_1 and y_2 are the roots of the auxiliary equation $px^2 - x + q = 0$. Thus $y_1 = 1$ and $y_2 = \frac{q}{p}$. (We have assumed in this section that $p \neq q$, so that these roots are distinct.) A particular solution of (2.11), which must be of the form Ck, is found by substitution to be $\frac{k}{q-p}$. Determination of the constants A_1, A_2 from the boundary conditions (2.12) gives

$$(2.13) \qquad D_k = \frac{N-k}{p-q} + \frac{q}{(p-q)^2}\left[\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^k\right]$$

and in particular the mean duration starting from the origin is given by

$$(2.14) \qquad E(X_o) = D_o = \frac{N}{p-q} + \frac{q}{(p-q)^2}\left[\left(\frac{q}{p}\right)^N - 1\right].$$

3. Variance of the Duration $(p \neq q)$

From (2.10), $D_o = E(X_o) = U_o'(1)$. By differentiating once more one sees that

$$(2.15) \qquad U_o''(1) = \sum_{n=0}^{\infty} n(n-1) u_{o,n} .$$

Moreover,

$$(2.16) \qquad \mathrm{Var}(X_o) = \sum_{n=0}^{\infty} (n - D_o)^2 u_{o,n} = \sum_{n=0}^{\infty} n^2 u_{o,n} - D_o^2$$

$$= U_o''(1) + D_o - D_o^2 .$$

The quantity $U_o''(1)$ can be found directly by differentiating in expression (2.9), or by methods similar to the above for finding $U_o'(1)$. Using the latter procedure, differentiating in (2.4) and (2.5), denoting $U_k''(1)$ by S_k and setting $s = 1$ we obtain the simultaneous recurrence relations

$$(2.17) \qquad \begin{aligned} S_k &= p S_{k+1} + q S_{k-1} + 2p D_{k+1} + 2q D_{k-1} = 0 \\ D_k &= p D_{k+1} + q D_{k-1} + 1 \end{aligned} .$$

The four boundary conditions are

$$(2.18) \qquad \begin{cases} p(S_o - S_1) = 2(D_o - 1) \\ p(D_o - D_1) = 1 \\ D_N = S_N = 0 \end{cases} .$$

Solving the system (2.17) (either by solving simultaneously for D_k and S_k or by substituting (2.12) and (2.11) into (2.17)) and then solving for S_k) we find the simple expression

$$(2.19) \qquad Var(X_o) = S_o + D_o - D_o^2 = \frac{q^2}{(p-q)^4}\left(\frac{q}{p}\right)^{2N}$$

$$+ \frac{4qN}{(p-q)^3}\left(\frac{q}{p}\right)^N + \frac{q(3-4q^2)}{(p-q)^4}\left(\frac{q}{p}\right)^N + \frac{4pqN}{(p-q)^3} - \frac{pq(4q+3)}{(p-q)^4}$$

4. Mean and Variance in the case $p = q = \frac{1}{2}$

The above formulae for $E(X_o)$ and $Var(X_o)$ have limits as $p \to \frac{1}{2}$ which give the correct values when $p = q = \frac{1}{2}$, but the computation involved is somewhat tedious, and it is just as easy to treat the case $p = q = \frac{1}{2}$ separately.

The recurrence relation for D_k now becomes

$$(2.20) \qquad D_k = \frac{1}{2}D_{k+1} + \frac{1}{2}D_{k-1} + 1$$

with boundary conditions

$$(2.21) \qquad D_N = 0 \, , \qquad D_o = D_1 + 2 \, .$$

The auxiliary equation $x^2 - 2x + 1 = 0$ has a double root 1. This means that solution of the corresponding homogeneous equation will be of the form $A_1 + A_2 k$ and that a particular solution of (2.20) will be of the form Ck^2. Proceeding as in section 2 we find

$$(2.22) \qquad D_k = N^2 + N - k - k^2 \, ,$$

whence

$$(2.23) \qquad E(X_o) = D_o = N^2 + N \, .$$

Finally, by the method of section 3, we find for the variance of duration in this case

$$(2.24) \qquad Var(X_o) = \frac{N(N+1)(2N^2+2N-1)}{3}$$

5. Asymptotic Distribution of Duration

In the preceding we have given exact expressions for the mean and variance of the duration X_o of the random walk, but not for the actual distribution of X_o. In certain applications it may be of importance to have at least an approximation to the complete distribution; moreover, many applications deal with random walks where N is large so that asymptotic results are just as valuable as exact ones. In this section we find the limiting distribution (in a sense to be made more precise) of X_o, using the results of sections 1-4; the results are relatively simple and, as will be indicated later, agree with what one might intuitively expect from the point of view of the theory of recurrent events (waiting times).

Since we are interested in asymptotic results, we consider the effect of increasing N while decreasing the time between steps.* (Increasing N alone will of course lead to a degenerate distribution.) We introduce a time parameter t to represent this time and wish to determine the limiting distribution of

*The motivation for the arguments in this section will be somewhat clearer if the reader is familiar with the mode of passage to a continuous limiting distribution from the geometric and Pascal distributions (cf. Feller [1] pp. 218-221).

duration as $N \to \infty$ and $t \to 0$. In order that this limiting distribution not be degenerate, we must impose some condition on the mode of passage to the limit; the most reasonable such condition is that the mean duration remain constant. As will be seen, this imposes a relationship between N and t so that they do not tend to their respective limits independently. As in section 1, we let $U_o(s)$ denote the generating function of the random variable X_o. Introducing the time parameter t, we consider the new random variable tX_o, which will take on possible values $0, t, 2t, \cdots, Nt$ with probabilities determined by $P\{tX_o = ta\} = P\{X_o = a\}$. The corresponding moment generating function of tX_o, which is by definition $E(e^{stX_o})$, is then given by $U_o(e^{st})$. In particular we note that

$$(2.25) \qquad \frac{d}{ds}[U_o(e^{st})] = tU_o'(1) ,$$

as it should, for $E(tX_o) = tE(X_o) = tU_o'(1)$. The moment generating function $U_o(e^{st})$ is, for fixed p, q and s, a function of two parameters, N and t, so that we will be interested in the limiting moment generating function $\phi(s)$, defined by

$$(2.26) \qquad \lim_{\substack{N\to\infty \\ t\to 0 \\ tE(X_o)\sim\lambda}} U_o(e^{st}) = \phi(s) ,$$

where λ is a positive constant. From the function $\phi(s)$, we hope to infer the form of the limiting distribution itself, using the continuity theorem and uniqueness properties of moment generating functions.

The three cases $p > q$ (drift to the right), $p < q$ (drift to the left) and $p = q = \frac{1}{2}$ (no drift) must be handled separately.

Case 1. Drift to the left $(p < q)$

From (2.14) we see that in this case the mean duration is asymptotically

$$D_o \sim \frac{q}{(p-q)^2} \left(\frac{q}{p}\right)^N . \tag{2.27}$$

Thus we require that $N \to \infty$ and $t \to 0$ in such a way as to make

$$\frac{q}{(p-q)^2} \left(\frac{q}{p}\right)^N t \sim \lambda . \tag{2.28}$$

Using (2.28) and (2.9), we have

$$\lim_{\substack{N\to\infty \\ t\to 0}} U_o(e^{st}) = \tag{2.29}$$

$$\lim_{\substack{N\to\infty \\ t\to 0}} \frac{2(2pe^{st})^N\, r(t)}{[1+r(t)]^N[1-2qe^{st}+r(t)] - [1-r(t)]^N[1-2qe^{st}-r(t)]}$$

where $r(t) = \sqrt{1-4pqe^{st}}$. It is understood that N and t here do not approach their respective limits independently, but rather in accordance with (2.28).

If we observe that $1-4pq = (q-p)^2$, that (2.28) implies $\lim_{\substack{N\to\infty \\ t\to 0}} Nt = 0$ and $\lim_{\substack{N\to\infty \\ t\to 0}} [1+0(t)]^N = 1$, and that the following hold as $t \to 0$:

(i) $e^{st} = 1 + st + 0(t^2)$

(ii) $\sqrt{1-4pqe^{st}} = \sqrt{1-4pq-8pqst+0(t^2)}$

$$= (1-4pq)^{1/2}[1 - \frac{8pqst}{1-4pq} + 0(t^2)]^{1/2}$$

$$= (q-p) - \frac{4pqst}{q-p} + 0(t^2) = q-p + 0(t)$$

(and from (i) and (ii))

(iii) $1 - 2qe^{st} + \sqrt{1-4pqe^{st}} = -\frac{2qst}{q-p} + 0(t^2)$

(iv) $1 - 2qe^{st} + \sqrt{1-4pqe^{st}} = 2(p-q) + 0(t)$

we find from (2.29) that

(2.30) $\lim_{\substack{N\to\infty \\ t\to 0}} U_o(e^{st})$

$$= \lim_{\substack{N\to\infty \\ t\to 0}} \frac{[2(q-p) + 0(t)] \cdot e^{stN}}{[\frac{2q}{2p} + 0(t)]^N(\frac{-2qst}{q-p} + 0(t^2)) - [\frac{2p}{2p} + 0(t)]^N(2(p-q) + 0(t))}$$

$$= \lim_{\substack{N\to\infty \\ t\to 0}} \frac{q-p}{\frac{q}{q-p}(\frac{q}{p})^N st - (p-q)}$$

$$= \frac{1}{1-\lambda s} .$$

Now $\frac{1}{1-\lambda s}$ is the moment generating function of the negative exponential distribution with mean λ ; that is, by the continuity theorem for moment generating functions, our limiting density function is

(2.31) $f(x) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}x}$ $(x > 0)$, where $\lambda = \frac{q}{(q-p)^2}[\frac{q}{p}]^N t$.

Note that the mean and variance of this distribution are λ and λ^2, in agreement with the asymptotic values of the mean and variance which we may obtain from (2.14) and (2.19).

<u>Case 2.</u> <u>Drift to the right</u> $(p > q)$

From (2.14), the mean duration is asymptotically

$$(2.32) \qquad D_o \sim \frac{N}{p-q}$$

and accordingly we require that $\lim\limits_{\substack{N\to\infty \\ t\to 0}} \frac{N}{p-q} t = \lambda$, with λ fixed.

Proceeding as before, we find from (2.32) and (2.9) that

$$(2.33) \qquad \lim_{\substack{N\to\infty \\ t\to 0}} U_o(e^{st}) = \frac{e^{\lambda(p-q)s}}{e^{-2q\lambda s}} = e^{s\lambda}$$

which is the moment generating function of the unitary distribution with mean λ. The reason for the approach to this degenerate distribution (all of whose probability is concentrated at the point λ) is that in introducing the time parameter t we have multiplied the mean duration by t and the variance by t^2; since the mean is $\frac{N}{p-q}t = \lambda$ and the variance, from (2.19), is asymptotically $\frac{4pq}{(p-q)^3}Nt^2$, the variance is then $\frac{4pq}{(p-q)^2}\lambda t$ which goes to zero at $t \to 0$.

More interesting from the point of view of applications is the manner of approach to the unitary distribution. Using arguments similar to those outlined for case 1 (noting that, in this case, $\sqrt{1-4pq} = p-q$ since $p > q$) it can be shown that

$$(2.34) \qquad U_0(e^{st}) \sim \frac{e^{\lambda(p-q)s}}{(1 - \frac{2q\lambda}{N} s)^N} \qquad (N \to \infty) \quad .$$

The denominator is the m.g.f. of a Γ-variate (Pearson Type III) distribution (the general density function of this family is of the form $Ax^{\lambda}e^{-\alpha x}$ where λ and α are non-negative constants, and $x > 0$); the moment generating function of a member of this family can be written in terms of the mean m and variance σ^2 as

$$(2.35) \qquad \frac{1}{(1 - \frac{\sigma^2}{m} s)^{m^2/\sigma^2}} \quad .$$

Thus $\frac{m^2}{\sigma^2} = N$ and $\frac{\sigma^2}{m} = \frac{2q\lambda}{N}$ so that the mean m of the curve represented by (2.32) is $2q\lambda$ and the variance is $\frac{4q^2\lambda^2}{N}$; the exponential factor $e^{\lambda(p-q)s}$ merely translates the curve to the right an amount $\lambda(p-q)$ so that the resultant mean is $\lambda(p-q) + 2q\lambda = \lambda$ as it should be; the effect of the shift is to truncate the curve at $\lambda(p-q) = Nt$ which corresponds to the fact that absorption at N is impossible for the first Nt time units. Note that if $m = \lambda$ and $\sigma^2 = \lambda^2$, (2.33) reduces to $\frac{1}{1-\lambda s}$ so that the exponential distribution is a special case of the Γ-variate.

<u>Case 3.</u> <u>No drift</u> $(p = q = \frac{1}{2})$

From (2.23), the mean in this case is asymptotically N^2, so we require that $\lim_{\substack{N\to\infty \\ t\to 0}} N^2 t = \lambda$. Setting $p = q = \frac{1}{2}$ in (2.9) and passing to the limit we have

$$\lim_{\substack{N\to\infty \\ t\to 0}} U_o(e^{st}) = \operatorname{sech}\sqrt{2s\lambda} \qquad s < 0 \tag{2.36}$$

(It is interesting to note, as a verification, that the variance as computed from (2.34) turns out to be $2\lambda^2/3$, in agreement with (2.24)).

From reference [6], page 257, the inverse Laplace Transform of sech $\sqrt{x}$ is

$$-\left[\frac{\partial}{\partial v}\theta_1\left(\frac{1}{2}v \mid i\pi t\right)\right]_{v=0} \tag{2.37}$$

where θ_1 is a theta-function given by

$$\theta_1(v \mid \tau) = (-i\tau)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-i\pi(v-\frac{1}{2}+n)^2/\tau} . \tag{2.38}$$

Carrying out the indicated operations, we find that the density function is

$$f(x) = \frac{1}{\sqrt{\pi}\left(\frac{x}{2\lambda}\right)^{3/2}} \sum_{n=-\infty}^{\infty} (-1)^n (2n-1)\, e^{-\frac{\lambda(2n-1)^2}{2x}} , \qquad (x > 0) \tag{2.39}$$

a result which is of the form one would obtain by the "method of images" [cf. especially [1], page 304, prob. 7 for a similar answer in the case of two absorbing barriers].

III. UNEQUAL STEPS $(r \neq \ell)$

We assume in this section that r and ℓ are any positive integers; without loss of generality we assume r and ℓ to be relatively prime, for if not the scale of the walk could be cut down by the common factor.

1. $pr \neq q\ell$

One can see directly by a physical argument similar to the one mentioned above (or by repeating the derivation of the generating function and subsequent arguments outlined in equation (2.1) - (2.12)) that the mean duration satisfies

$$(3.1) \qquad D_k = pD_{k+r} + qD_{k-\ell} + 1 \qquad\qquad k = +\ell, +\ell+1, \cdots, N-r$$

with boundary conditions

$$(3.2) \qquad \begin{cases} D_k = pD_{k+r} + qD_o + 1 & k = 0, 1, \cdots, \ell \\ D_k = qD_{k-\ell} + 1 & k = N-r+1, \cdots, N-1 \\ D_N = 0 \ , & \end{cases}$$

Equations (3.2) represent $r+\ell$ boundary conditions to be imposed on any solution to the non-homogeneous equation (of order $r+\ell$)(3.1) and hence determine a unique solution. As is seen by substitution into (3.1), the conditions (3.2) may be replaced by the simpler conditions

$$(3.3) \qquad \begin{cases} D_N = D_{N+1} = \cdots = D_{N+r-1} = 0 \\ D_{-\ell} = D_{-\ell+1} = \cdots = D_{-1} = D_o \end{cases}$$

Note that since D_o is undetermined, (3.3) specifies exactly $r+\ell$ conditions, as it should.

The auxiliary equation of (3.1) is

$$(3.4) \qquad px^{r+\ell} - x^{\ell} + q = 0$$

which has $r+\ell$ roots $1, x_1, x_2, \cdots, x_{r+\ell-1}$. Let

$$\phi(x) = px^{r+\ell} - x^{\ell} + q \tag{3.5}$$

Simple considerations establish the following facts concerning the function $\phi(x)$ and roots of the equation $\phi(x) = 0$.

(3.6)

(a) All the roots are simple.

(b) There are exactly two positive roots, say 1 and x_1, with

case 1. $x_1 > 1$ if $pr - q\ell < 0$

case 2. $x_1 < 1$ if $pr - q\ell > 0$

(c) $\phi(x) < 0$ for all real x between 1 and x_1

(d) The absolute value ρ of any complex or negative root satisfies $\phi(\rho) > 0$. For let x_i be the root; $|x_i| = \rho$. Then $px_i^{r+\ell} - x_i^{\ell} + q = 0 \Rightarrow x_i^{\ell} = px_i^{r+\ell} + q$ and by the triangle inequality $\rho^{\ell} < p\rho^{r+\ell} + q$ (note that strict inequality holds since x_i is not positive real) which is equivalent to $\phi(\rho) = p\rho^{r+\ell} - \rho^{\ell} + q > 0$

(e) From (d) it follows that there are no roots in the open annuli

$$\begin{cases} 1 < |z| < x_i & (pr - q\ell < 0) \\ x_1 < |z| < 1 & (pr - q\ell > 0); \end{cases}$$

in fact 1 and x_1 are the only roots in the closure of these annuli.

The conclusion of the following lemma concerning the roots of $\phi(x) = 0$ will be of importance later:

Lemma

Exactly ℓ of the roots of $\phi(x) = 0$ are in absolute value less than or equal to 1 if $pr - q\ell < 0$ and less than or equal to $x_1 < 1$ if $pr - q\ell < 0$.

For simplicity, we prove the lemma for $pr - q\ell < 0$; the proof in the other case is similar.

Proof :

Let C be a simple closed contour contained in the annulus $1 < |z| < x_1$. Let $f(z) = -z^{\ell}$ and $g(z) = pz^{r+\ell} + q$. Then $f(z) \neq 0$ on C and $|g(z)| < |f(z)|$ on C since by (c) of (3.6) $p|z|^{r+\ell} - |z|^{\ell} + q < 0$ on C whence $|f(z)| = |z|^{\ell} > p|z|^{r+\ell} + q > |pz^{r+\ell} + q| = |g(z)|$. Thus by Rouché's theorem, the number of zeros within C of $f(z)$ is equal to the number of zeros within C of $f(z) + g(z) = \phi(z)$. But $f(z)$ has a zero of multiplicity ℓ (counted as ℓ zeros) within C ; hence $\phi(z)$ has ℓ zeros within C, since by (a) of (3.6) all the zeros of $\phi(z)$ are simple. Since the annulus $1 < |z| < x_1$ is free of zeros (by (d) of (3.6)) there are ℓ zeros on or within the circle $|z| = 1$. (In fact $\ell - 1$ of them lie within, and the zero 1 lies on the circle). Q.E.D.

For the reader's convenience we include the following:

Rouché's Theorem :

Suppose $f(z)$ and $g(z)$ are analytic on and within a simple closed contour C of the complex plane, and that on C $f(z) \neq 0$ and $|g(z)| < |f(z)|$. Then $f(z)$ and $f(z) + g(z)$

have the same number of zeros within C .

Returning now to the problem of finding a solution to equation (3.1), we note that a particular solution is $\frac{k}{q\ell - pr}$; hence the complete solution to (3.1) is of the form

$$(3.7) \qquad D_k = \frac{k}{q\ell - pr} + \sum_{j=1}^{r+\ell-1} a_j x_j^k + a_o$$

where the $r+\ell$ constants a_j are determined by conditions (3.3).

Using these conditions we now seek to obtain an approximation to D_o . Substituting in (3.7) we obtain

$$(3.8) \qquad D_o = a_o + a_1 + \cdots + a_{r+\ell-1}$$

and

$$(3.9) \qquad 0 = \frac{N}{q\ell - pr} + \sum_{j=1}^{r+\ell-1} a_j x_j^N + a_o$$

and finally,

$$(3.10) \qquad \sum_{j=1}^{r+\ell-1} a_j x_j^k + \frac{k}{q\ell - pr} = \sum_{j=1}^{r+\ell-1} a_j x_j^{k+1} + \frac{k+1}{q\ell - pr}$$

$$k = \begin{Bmatrix} \ell, -\ell+1, \cdots, -3, -2, -1 \\ N, N+1, N+2, \cdots, N+r-2 \end{Bmatrix} .$$

Collecting terms in (3.10) we have

$$(3.11) \qquad \sum_{j=1}^{r+\ell-1} a_j x_j^k (1 - x_j) = \frac{1}{q\ell - pr} \qquad k = \begin{Bmatrix} -\ell, \cdots, -1 \\ N, N+1, \cdots, N+r-2 \end{Bmatrix} .$$

The equations (3.11) are $r+\ell-1$ linear equations in the unknown $a_1, \cdots, a_{r+\ell-1}$. In terms of these, it can be seen from (3.8) and (3.9) that the desired solution D_o is then

$$(3.12) \qquad D_o = \frac{N}{pr-q\ell} + \sum_{j=1}^{r+\ell-1} a_j(1-x_j^N) \quad .$$

In the sequel we will show that for N large the root x_1 is the only root which is of importance to the size of the right hand member of (3.12) in case $pr-q\ell < 0$ and that the term $\frac{N}{pr-q\ell}$ is the only term of importance in case $pr-q\ell > 0$. We first remark that D_o must be real, from physical considerations, this could be verified directly using the fact that the complex roots occur in conjugate pairs. Rather than express (3.12) in terms of real quantities only, we will continue to express the roots and constants a_j in terms of complex numbers, and will use the fact that D_o is real in the final results.

The determinant of the system (3.11) has the form

$$(3.13) \quad \begin{vmatrix} x_1^{-1}(1-x_1) & x_2^{-1}(1-x_2) & \cdots & x_{r+\ell-1}^{-1}(1-x_{r+\ell-1}) \\ x_1^{-2}(1-x_1) & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ x_1^{-\ell}(1-x_1) & x_2^{-\ell}(1-x_2) & & x_{r+\ell-1}^{-\ell}(1-x_{r+\ell-1}) \\ x_1^{N}(1-x_1) & x_2^{N}(1-x_2) & & \vdots \\ \vdots & \vdots & & \vdots \\ x_1^{N+r-2}(1-x_1) & x_2^{N+r-2}(1-x_2) & & x_{r+\ell-1}^{N+r-2}(1-x_{r+\ell-1}) \end{vmatrix} .$$

Noticing that $\prod_{i=1}^{r+\ell-1} (x_i-1)$ is the product of the roots of the equation $\frac{\phi(x+1)}{x} = 0$, and is thus $(-1)^{r+\ell-1}$ times the constant term in $\frac{\phi(x+1)}{x}$ we have

$$\prod_{i=1}^{r+\ell-1} (1-x_i) = \frac{pr-q\ell}{p} \quad . \tag{3.14}$$

Using (3.13), elementary manipulations show that the above determinant reduces to the form $\frac{pr-q\ell}{p} \Delta$ where

$$\Delta = \begin{vmatrix} x_1^{-1} & x_2^{-1} & \cdots & x_{r+\ell-1}^{-1} \\ x_1^{-2} & x_2^{-2} & & \cdot \\ \vdots & \vdots & & \vdots \\ x_1^{-\ell} & x_2^{-\ell} & & x_{r+\ell-1}^{-\ell} \\ x_1^{N} & x_2^{N} & & x_{r+\ell-1}^{N} \\ \vdots & \vdots & & \vdots \\ x_1^{N+r-2} & x_2^{N+r-2} & & x_{r+\ell-1}^{N+r-2} \end{vmatrix} \quad . \tag{3.15}$$

Similarly, the determinant formed by replacing the i^{th} column of (3.13) by a column of constants $\frac{1}{pr-q\ell}$ has the form

$$\frac{1}{p(1-x_i)} \Delta_i \tag{3.16}$$

where Δ_i is the determinant Δ with the i^{th} column replaced by a column of 1's.

The constant a_i is then given by

$$(3.17) \qquad a_i = \frac{1}{(pr-q\ell)(1-x_i)} \frac{\Delta_i}{\Delta}$$

and the solution (3.12) becomes

$$(3.18) \qquad D_o = \frac{N}{pr-q\ell} + \sum_{j=1}^{r+\ell-1} c_j \frac{\Delta_i}{\Delta}(1-x_j^N)$$

where $c_j = \frac{1}{(pr-q\ell)(1-x_i)}$.

From (3.15) it can be seen that the determinant Δ is a linear combination of products of the numbers $x_1, x_2, \cdots, x_{r+\ell-1}$ raised to the powers $-1, -2, \cdots, -\ell$ and $N, N+1, \cdots, N+r-2$, each exponent occurring once and only once in each term. In each term, $r-1$ of the factors involve an exponent with an N in it; Δ can then be expressed as a linear combination of N^{th} powers of products of the numbers $x_1, \cdots, x_{r+\ell-1}$ taken $r-1$ at a time. The leading term of Δ is, for large N, that term involving the $r-1$ largest roots (in absolute value). Let $x_2, x_3, \cdots, x_r$ be these; then

$$(3.19) \qquad |\Delta| \sim K_1(|x_2||x_3|\cdots|x_r|)^N$$

where K_1 is some constant.

By a similar argument, reference to (3.16) and the definition of Δ_i shows that Δ_i may be expressed as a linear combination of products of the numbers $x_1, x_2, \cdots, x_{i-1}$, $x_{i+1}, \cdots, x_{r+\ell-1}$ taken $(r-1)$ at a time and $(r-2)$ at a time (as above, each factor raised to one of the powers

$-\ell, \cdots, -1, N, \cdots, N+r-2$); note that the root x_i does not occur in Δ_i .

Again, the leading term is that involving the r-1 largest roots (in absolute value). Since the missing root x_i may be among the r-1 largest, it will be convenient to consider the quantity $\Delta_i(1-x_i^N)$. This is also a linear combination of terms of the type described above; the leading term is that involving the r largest roots. (Those terms arising from $\Delta_i x_i^N$ involve products of all the roots taken r at a time). By the preceding lemma and (e) of (3.6), it follows that for $pr-q\ell < 0$ there are r-1 roots larger than x_1 and ℓ smaller; thus x_1 is exactly r^{th} in size and

$$(3.20) \qquad |\Delta_i(1-x_i^N)| \sim K_2(|x_1||x_2|\cdots|x_r|)^N$$

and it follows that

$$(3.21) \qquad |\frac{\Delta_i}{\Delta}| \sim K_3|x_1|^N = K_3x_1^N \quad .$$

Finally, since $x_1 > 1$ the term $\frac{N}{pr-q\ell}$ is of smaller order than x_1^N , we have from (3.18)

$$(3.22) \qquad D_o \sim Kx_1^N \qquad\qquad (pr-q\ell < 0)$$

where K is some real constant. (Here we use the fact that D_o is real).

For $pr-q\ell > 0$ the argument is the same with the exception that x_1 is everywhere replaced by 1 which by the lemma is r^{th} in size. Thus $|\frac{\Delta_i}{\Delta}| \sim K_3 1^N = K_3$ so that

(3.23) $$D_o \sim \frac{N}{pr-q\ell} \qquad (pr-q\ell > 0)$$

2. <u>The case $pr = q\ell$</u> .

The lemma as stated holds only in case $pr \neq q\ell$; it asserts that if $pr-q\ell < 0$ $r-1$ roots are greater than x_1 in absolute value and that if $pr-q\ell > 0$ $r-1$ roots are greater than 1 in absolute value. As $pr \to q\ell$, $x_1 \to 1$ so that in the limit $(pr = q\ell)$ there is a double root at 1 ; it can easily be shown that this is the only positive root. The method used in proving the lemma no longer works in this case since the annulus in which the curve C was inscribed ceases to exist. However, continuity arguments* show that in the limit as $pr \to q\ell$ the number of roots on or exterior to the unit circle must always exceed $r-1$. Using this fact we are in position to extend the preceding argument to the present case.

Since 1 is a double root of (3.4) a new particular solution to (3.1) must be found; it is easily verified that

(3.24) $$-\frac{k^2}{r\ell}$$

is such a solution, so that (3.7) is replaced by

(3.25) $$D_k = -\frac{k^2}{r\ell} + \sum_{j=1}^{r+\ell-2} a_j x_j^k + a_o + a_1 k \ .$$

Following the previous arguments it then results that

(3.26) $$D_o = \frac{N^2}{r\ell} + \sum_{j=2}^{r+\ell-1} a_j(1-x_j)^N + a_1 N$$

* The authors are indebted to O. S. Rothaus who suggested a proof.

where again the a_j are given by quotients of determinants of form similar to those of the preceding section. Using the remark that at least r roots are in absolute value greater than or equal to 1, it follows from the same type of argument used before that

$$(3.27) \qquad D_o \sim \frac{N^2}{r\ell} \quad .$$

IV. COMPARISON OF RESULTS

For convenience we restate the principal results from sections II and III concerning the mean duration D_o :

$$(4.1) \qquad D_o = \frac{N}{p-q} + \frac{q}{(p-q)^2}[(\frac{q}{p})^N - 1] \qquad (r = \ell = 1, q \neq p)$$

From (4.1) we have

$$(4.2) \qquad D_o \sim \frac{q}{(p-q)^2}[\frac{q}{p}]^N \qquad (r = \ell = 1, p-q < 0)$$

$$(4.3) \qquad D_o \sim \frac{N}{p-q} \qquad (r = \ell = 1, p-q > 0).$$

Also

$$(4.4) \qquad D_o = N^2 + N \qquad (r = \ell = 1, p = q)$$

so that

$$(4.5) \qquad D_o \sim N^2 \qquad (r = \ell = 1, p = q).$$

Now in the general case

$$(4.6) \qquad D_o \sim Kx_1^N \qquad (pr - q\ell < 0)$$

(4.7) $D_o \sim \frac{N}{pr-q\ell}$ $(pr-q\ell > 0)$

and finally

(4.8) $D_o \sim \frac{N^2}{r\ell}$ $(q\ell = pr)$.

One sees immediately that (4.5) and (4.8) agree in case $r = \ell = 1$ and that so do (4.3) and (4.7). To unify the formulae we note that the mean m and variance σ^2 of the individual steps are $pr-q\ell$ and $pq(r+\ell)^2$ respectively, so that in general for $m > 0$ (drift to the right) $D_o \sim \frac{N}{m}$. For $m < 0$ the comparison is not as apparent; (4.2) and (4.6) agree in form and $x_1 = \frac{q}{p}$ for $r = \ell = 1$; but to complete the comparison we need to express x_1 in the general case as a function of p, q, r and ℓ . By analogy with the theory of birth and death processes in which the ratio of birth rate to the death rate plays a central role (cf. [1] page 374), one is led to conjecture $\frac{q\ell}{pr}$ as a generalization of $\frac{q}{p}$. However, consideration must be given the fact that the size of the walk N should be normalized to proper dimensions since ℓ and r exceed unity; a logical normalization is one that reduces the standard deviation of an individual step in the general case to that of the case $r = \ell = 1$. The ratio of standard deviations in the two cases is $\frac{\sqrt{pq}\,(r+\ell)}{2\sqrt{pq}} = \frac{(r+\ell)}{2}$, so that N should be divided by this factor where it appears. This gives $D_o \sim [\frac{q\ell}{pr}]^{\frac{2N}{(r+\ell)}}$ whence $x_1 = [\frac{q\ell}{pr}]^{\frac{2}{r+\ell}}$. Numerical trials show that this is a remarkably good approximation of the positive root x_1, especially if r and ℓ are nearly equal and p is not too close to 0 or 1.

As an additional remark, we note that for $r = \ell = 1$ the constant K in $D_o \sim K x_1^N$ is determined as $\frac{q}{(p-q)^2}$. Although the constant for the general case is a rather complicated expression involving products of the $r+\ell$ roots of equation (3.4), some attempts at simplification and estimation of the constant have shown that the largest (and possibly the only significant) term in the expression for the constant is $\frac{x_1}{(pr-q\ell)(1-x_1)}$.

Since this reduces to $\frac{q}{(p-q)^2}$ in case $r = \ell = 1$ (and $x_1 = \frac{q}{p}$) it is a tempting surmise that it is a reasonable approximation for K. We would then have

$$(4.9) \qquad D_o \sim \frac{1}{(pr-q\ell)(1-[\frac{q\ell}{pr}]^{2/\ell+r})} [\frac{q\ell}{pr}]^{\frac{2(N+1)}{r+\ell}} \qquad (pr-q\ell < 0)$$

which is exact for $r = \ell = 1$ and approximate for general r and ℓ.

We note that for $m > 0$ our results (4.3) and (2.19) agree exactly with those given by Bartlett (cf. [4] page 20, eq. (22)) for the random walk with single absorbing barrier at N and no restrictions on the left. That the results are identical is not surprising since for with a drift to the right ($m > 0$) a restraining barrier at the origin is of virtually no signifance to the duration of the walk. It is interesting to note that the results given by Bartlett hold for the general r and ℓ; in terms of the mean m and variance σ^2 they are $D_o \sim \frac{N}{m}$ and $V_o \sim \frac{N\sigma^2}{m^3}$. Although we have not given the variance for the general case, the agreement above leads to the conjecture that

$$(4.10) \qquad V_o \sim \frac{Npq(r+\ell)^2}{(pr-q\ell)^2} \qquad (pr-q\ell > 0) \quad .$$

Since we do not have the generating function for duration in the general case, we cannot repeat the steps of section II part 5 to find asymptotic limiting distributions. However, the following heuristic discussion may, we hope, furnish a plausible conjecture regarding the limiting distributions in general.

It is a result of the theory of recurrent events that the distribution of the time up to the first event is exponential if the event occurs at random in time (i.e., occurrence in a small interval of time is proportional to the length of the interval, and independent of the position of the interval in time); furthermore the distribution of the time until the r^{th} event is Γ-variate (with suitable parameters). These distributions are used in many connections, especially in queueing theory and have recently been called "Erlangian distributions" by Kendall. That these distributions arise in section 5 of II is not surprising for

(a) with a drift to the left, the event (absorption at N) does occur "approximately" at random in time; that is, the walk tends to persist at or near the origin and once at the

the origin, what happens to the process in the future is independent of its past history. Thus we should expect the duration to be approximately exponential.

(b) with a drift to the right, the event (absorption at N) takes place only after a succession of events, each of which consist of the walk moving to the right by an amount $\frac{N}{m}$ (where m is the size of the average step), and so we could expect the limiting distribution of duration to be of Γ-variate type.

Now statements (a) and (b), while heuristic, are justified rigorously in case $r = \ell = 1$ by the results of section II part 5. Since these statements require only a drift to the left or right, it is a tempting conjecture that the same limiting distributions obtain in the general case. This conjecture is made more appealing by the manner in which the mean duration in the general case (section III) turns out to be a simple generalization of the mean for the special case $r = \ell = 1$.

BIBLIOGRAPHY

1. Feller "Probability Theory and its Applications" Chapter XIV

2. Uspensky "Introduction to Mathematical Probability" Chapters V and VII

3. Lobachevsky "Analytic and Algebraic Topology of Locally Euclidean Metrizations of Infinitely Differentiable Remannian Manifolds" (Unpublished)

4. Bartlett "Stochastic Processes. Methods and Applications" Chapter II

5. Kemperman "The General One-Dimensional Random Walk with Absorbing Barriers" (Ph.D. Dissertation, Amsterdam 1950)

6. Erdélyi, Magnus, Oberhettinger, Tricomi "Tables of Integral Transforms" Volume I of Bateman Manuscript Project